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Nonlinear internal gravity waves in a rotating fluid

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The interaction between internal gravity waves in a rotating frame and the mean flow is discussed for the case when the properties of the mean flow vary slowly on a scale determined by the local wave structure. The principle of conservation of wave action is established. It is shown that the main effect of the waves on the Lagrangian mean velocity is due to an appropriate 'radiation stress' tensor. A circulation theorem and a potential-vorticity equation are derived for the mean velocity.

1. Introduction

The interaction between internal gravity waves and the mean velocity (when the length scale of the waves is much shorter than that of the mean flow) was first examined in detail by Bretherton (1969) for small amplitude waves. Subsequently, this work was extended to finite amplitude waves, incorporating the perturbation effects of friction and compressibility, by Grimshaw (1972, 1974). The purpose of this paper is to discuss this interaction in a rotating frame. Two significant differences emerge. First, because of the rotation, the mean velocity is constrained to be in approximate geostrophic balance and departures from this balance are determined by a 'radiation stress' tensor, derived from the waves. This tensor is just the local average over the waves of the particle displacement with the pressure gradient. Second, the equations for the mean motion adopt their simplest form when expressed in terms of Lagrangian mean velocities (cf. Bretherton 1971); indeed, the equations for the mean Eulerian velocity contain buoyancy flux terms due to the waves, as well as Reynolds stresses. However, when Lagrangian mean velocities are introduced, these combine to form a single 'radiation stress' tensor.

In § 2, the concept of a modulated wave is defined, the Eulerian mean equations are derived and the conservation of wave action is established. In § 3 the Lagrangian mean velocity is introduced and the equations for the mean motion simplified. In § 4 the transport equations (i.e. the equations for the mean motion plus the equation for the conservation of wave action) are discussed, a circulation theorem is established and it is shown that the effect of the waves on the mean flow may be described by a forcing term in the potential-vorticity equation.

We shall complete this section with a description of the terminology and the equations of motion. Let L_1 be a length scale characterizing the wavelength and let N_1^{-1} be the time scale, where N_1 is a typical value of the Brunt–Väisällä frequency. Then

$$\epsilon = N_1^2 L_1/g \tag{1.1}$$

is a small parameter, being the ratio of a typical wavelength to the length scale of the mean density profile; it will be assumed that ϵ also characterizes the ratio of a typical wavelength to the length scale of the mean velocity field. If c_1 is a typical value of the speed of sound, then

$$F = gL_1/c_1^2 \tag{1.2}$$

is also a small parameter, being a ratio of L_1 to the 'scale height' of the atmosphere, and indicative of the effects of compressibility. The ratio F/ϵ is a property of the mean state of the fluid; for an isothermal ideal gas $F/\epsilon = (\gamma-1)^{-1}$, where γ is the ratio of the specific heats; it will be assumed that F is $O(\epsilon)$. If μ_1 is a typical value of the viscosity, then the third relevant small parameter is

$$E = \mu_1/\rho_1 N_1 L_1^2, \tag{1.3}$$

where ρ_1 is a typical value of the density; E measures the frictional effects and it will be assumed that E is $O(\epsilon)$. The velocity scale will be $N_1 L_1$ and the pressure scale will be $\rho_1 g L_1/\epsilon$ (the hydrostatic scale for the mean state).

The equations governing the conservation of mass, momentum and entropy, referred to a frame rotating with angular velocity $\mathbf{\Omega}$, are, respectively, using non-dimensional variables,

$$d\rho/dt + \rho\nabla\,.\,\mathbf{u} = 0, \tag{1.4}$$

$$\rho\, d\mathbf{u}/dt + \rho 2\mathbf{\Omega}\times\mathbf{u} + \epsilon^{-2}\nabla p + \epsilon^{-1}\rho\mathbf{k} = E\mu\nabla^2\mathbf{u} + \ldots, \tag{1.5}$$

$$\frac{d\rho}{dt} - \frac{F}{\epsilon}\frac{1}{c^2}\frac{dp}{dt} = \sigma E k \nabla^2\rho + \ldots. \tag{1.6}$$

Here $\mathbf{u}$ is the velocity relative to the rotating frame, p is the pressure, ρ the density, c the speed of sound (a prescribed function of p and ρ), and $\mathbf{k}$ is a unit vector in the vertical (z) direction; μ is the viscosity, k the thermal diffusivity (both prescribed functions of p and ρ) and σ is a constant, the Prandtl number, being the ratio of a typical value of the thermal diffusivity to a typical value of the viscosity. The terms omitted on the right-hand sides of (1.5) and (1.6) are $O(\epsilon)$ compared with the terms which have been retained. The centrifugal effects of rotation have been absorbed into the gravitational term ($g\mathbf{k}$ in dimensional variables), which is assumed to be a constant. The x and y axes will be in the easterly and northerly directions respectively; thus

$$2\mathbf{\Omega} = (0, 2\Omega_H, 2\Omega_V). \tag{1.7}$$

Note that $2\Omega_H$ and $2\Omega_V$ are the non-dimensional horizontal and vertical components of the Coriolis parameter, which has been scaled by N_1.

2. Modulated waves

We shall show that the equations of motion possess an asymptotic solution which is locally a plane sinusoidal wave but whose properties vary on length and time scales $O(\epsilon^{-1})$. The procedure used is similar to that developed by Grimshaw (1974) for internal gravity waves in the absence of rotation. Thus let

$$\mathbf{X} = \epsilon\mathbf{x}, \quad T = \epsilon t, \tag{2.1}$$

and seek a solution of the form

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{V}(\mathbf{X},T;\epsilon)+\mathbf{v}(\theta;\mathbf{X},T;\epsilon),\\ \rho &= R(\mathbf{X},T;\epsilon)\{1+\epsilon r(\theta;\mathbf{X},T;\epsilon)\},\\ p &= Q(\mathbf{X},T;\epsilon)+\epsilon^2 q(\theta;\mathbf{X},T;\epsilon),\end{aligned}\right\} \tag{2.2}$$

where the phase θ is defined to be that the local frequency $\omega = -\theta_t$ and the local wavenumber $\boldsymbol{\varkappa} = \nabla_{\mathbf{x}}\theta$ are functions of $(\mathbf{X}, T)$ and so are slowly varying. Thus

$$\theta = \epsilon^{-1}\Theta(\mathbf{X},T;\epsilon) \tag{2.3}$$

and

$$\omega = -\Theta_T, \quad \boldsymbol{\varkappa} = \nabla\Theta. \tag{2.4}$$

(In this and subsequent sections all spatial and time derivatives are with respect to $\mathbf{X}$ and T.) $\mathbf{v}$, r and q are periodic in θ with period 2π and have zero mean; thus $\mathbf{V}$, R and Q are the mean velocity, density and pressure respectively. All these variables are $O(1)$ with respect to ϵ, and are assumed to possess asymptotic power-series expansions in ϵ. Note that it has been anticipated that the density and pressure fluctuations will be $O(\epsilon)$ and $O(\epsilon^2)$ respectively (the appropriate scaling for internal gravity waves).

Substitution of (2.2) into (1.4)–(1.6) gives

$$\boldsymbol{\varkappa}.\mathbf{v}_\theta+\epsilon\nabla.\mathbf{v}+\epsilon\nabla.\mathbf{V}+\frac{\epsilon}{R}\frac{DR}{DT}+\frac{\epsilon\mathbf{v}.\nabla R}{R}+\frac{\epsilon}{1+\epsilon r}\frac{dr}{dt} = 0, \tag{2.5}$$

$$\begin{aligned} &R(1+\epsilon r)\{\epsilon D\mathbf{V}/DT+\epsilon\mathbf{v}.\nabla\mathbf{V}+d\mathbf{v}/dt+2\boldsymbol{\Omega}\times\mathbf{V}+2\boldsymbol{\Omega}\times\mathbf{v}\}\\ &\qquad +\epsilon^{-1}\nabla Q+\boldsymbol{\varkappa}q_\theta+\epsilon\nabla q+\epsilon^{-1}R\mathbf{k}+Rr\mathbf{k} = E\bar{\mu}\kappa^2\mathbf{v}_{\theta\theta}+\ldots, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\frac{1}{R}\frac{DR}{DT}+\frac{\mathbf{v}.\nabla R}{R}+\frac{1}{1+\epsilon r}\frac{dr}{dt}-\frac{F}{\epsilon RC^2}\left\{\frac{DQ}{DT}+\mathbf{v}.\nabla Q+\epsilon\frac{dq}{dt}\right\}\\ &\qquad -F\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho c^2}\right)\right]Rr\left\{\frac{DQ}{DT}+\mathbf{v}.\nabla Q\right\} = \sigma E\bar{k}\kappa^2 r_{\theta\theta}+\ldots, \end{aligned} \tag{2.7}$$

where

$$\frac{d}{dt} \equiv -\omega^*\frac{\partial}{\partial\theta}+\mathbf{v}.\boldsymbol{\varkappa}\frac{\partial}{\partial\theta}+\epsilon\frac{D}{DT}+\epsilon\mathbf{v}.\nabla, \tag{2.8}$$

$$D/DT \equiv \partial/\partial T+\mathbf{V}.\nabla, \tag{2.9}$$

$$\omega^* = \omega-\boldsymbol{\varkappa}.\mathbf{V}, \quad \text{the intrinsic frequency}, \tag{2.10}$$

and the omitted terms are $O(\epsilon^2)$; d/dt is the (exact) time derivative following a fluid particle. Here κ denotes the magnitude of $\boldsymbol{\varkappa}$. In (2.7) C denotes c evaluated at (Q, R) and the differentiation of $(\rho c^2)^{-1}$ is with respect to ρ at constant pressure p. In (2.6) and (2.7), $\bar{\mu}$ and $\bar{k}$ denote μ and k evaluated at (Q, R). Equations (2.5)–(2.7) may now be averaged with respect to the phase θ over the period 2π. Let angular brackets denote averages:

$$\langle f\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\theta;\mathbf{X},T;\epsilon)\,d\theta \tag{2.11}$$

for any variable f. Then, after some manipulation, it follows that

$$DR/DT+R\nabla.\mathbf{V}+\epsilon\nabla.\langle Rr\mathbf{v}\rangle = 0, \tag{2.12}$$

$$\epsilon R\,D\mathbf{V}/DT+R2\boldsymbol{\Omega}\times\mathbf{V}+\epsilon\nabla.\langle R\mathbf{v}\mathbf{v}\rangle+\epsilon 2\boldsymbol{\Omega}\times\langle Rr\mathbf{v}\rangle+\epsilon^{-1}\nabla Q+\epsilon^{-1}R\mathbf{k} = O(\epsilon^2), \tag{2.13}$$

$$\frac{DR}{DT}+\epsilon\nabla.\langle Rr\mathbf{v}\rangle-\frac{F}{\epsilon C^2}\frac{DQ}{DT} = O(\epsilon^2). \tag{2.14}$$

Note that (2.12) is exact. These equations are most simply derived by first rewriting (1.4)–(1.6) in conservation form, then substituting (2.2) and averaging. For example, this procedure shows that (2.5) can be written in the form

$$-\epsilon\omega^* R r_\theta + R\{(1+\epsilon r)\,\boldsymbol{\kappa}.\mathbf{v}\}_\theta + \epsilon R_T + \epsilon^2 (Rr)_T + \epsilon\nabla.\{R(1+\epsilon r)(\mathbf{V}+\mathbf{v})\} = 0,$$

and averaging this equation produces (2.12). It follows from (2.13) that, to $O(\epsilon^2)$, the mean state is hydrostatic and geostrophic; the horizontal gradients of Q and R are also $O(\epsilon)$ smaller than the vertical gradients. It will be assumed below that Q and R are, to $O(\epsilon)$, functions of Z only; then it follows that, again to $O(\epsilon)$, $\mathbf{V}$ is horizontal and non-divergent. [It has been pointed out by Garrett (1968) that if Q and R depend on both Z and T to leading order in ϵ then there will be an $O(1)$ vertical velocity, likewise depending on both Z and T. This rather artificial case is not considered here, although it may be shown that the result (2.36) for the wave action is still valid for this case (cf. Grimshaw (1975*b*), where this result was derived for the linearized system).]

The averaged equations (2.12)–(2.14) may now be subtracted from their respective counterparts (2.5)–(2.7) to yield the following equations for the fluctuating variables:

$$\boldsymbol{\kappa}.\mathbf{v}_\theta = -\epsilon I_\theta, \tag{2.15}$$

$$-\omega^*\mathbf{v}_\theta + 2\boldsymbol{\Omega}\times\mathbf{v} + R^{-1}q_\theta\boldsymbol{\kappa} + r\mathbf{k} = -\epsilon\mathbf{F}, \tag{2.16}$$

$$-\omega^* r_\theta - N^2 w = -\epsilon H, \tag{2.17}$$

where

$$N^2 = -\frac{R_Z}{R} - \frac{F}{\epsilon}\frac{1}{C^2}, \quad w = \mathbf{v}.\mathbf{k}. \tag{2.18), (2.19}$$

N is the Brunt–Väisälä frequency, and to leading order in ϵ is a function of Z alone. The right-hand sides of these equations are $O(\epsilon)$, and the specific expressions for I_θ, $\mathbf{F}$ and H are, after some manipulation,

$$I_\theta = \nabla.\mathbf{v} - \frac{F}{\epsilon}\frac{w}{C^2} + O(\epsilon), \tag{2.20}$$

$$\begin{aligned}\mathbf{F} = {}& D\mathbf{v}/DT + \mathbf{v}.\nabla\mathbf{V} + \nabla.\{(R\mathbf{v}\mathbf{v}) - \langle R\mathbf{v}\mathbf{v}\rangle\} \\ & - (I\mathbf{v})_\theta - \omega^*(r\mathbf{v})_\theta + r2\boldsymbol{\Omega}\times\mathbf{V} + 2\boldsymbol{\Omega}\times\{r\mathbf{v} - \langle r\mathbf{v}\rangle\} \\ & + R^{-1}\nabla q - \epsilon^{-1}R^{-1}E\bar{\mu}\kappa^2\mathbf{v}_{\theta\theta} + O(\epsilon),\end{aligned} \tag{2.21}$$

$$\begin{aligned}H = {}& \frac{Dr}{DT} + \frac{\mathbf{v}_H.\nabla_H R}{\epsilon R} + \frac{\nabla.(Rr\mathbf{v} - \langle Rr\mathbf{v}\rangle)}{R} - (Ir)_\theta \\ & + \frac{F}{\epsilon}\frac{1}{RC^2}(2\boldsymbol{\Omega}\times\mathbf{V}.\mathbf{v} + \omega^* q_\theta) + \frac{F}{\epsilon}\overline{\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho c^2}\right)\right]}R^2 rw - \epsilon^{-1}\sigma E\bar{k}\kappa^2 r_{\theta\theta} + O(\epsilon).\end{aligned} \tag{2.22}$$

Here and subsequently subscript H denotes a horizontal component. The constant of integration in the expression for I is chosen such that I has zero mean. In deriving these expressions, use has been made of the zero-order relations between $\mathbf{v}$, r and q obtained by replacing the right-hand sides of (2.15)–(2.17) with zeros; this is justified as I, F and H are used below only after they have been evaluated to lowest order in ϵ. Some use has also been made of the fact that Q and R are functions of Z only, to leading order in ϵ, and so, for example, $\nabla_H R$ is $O(\epsilon)$ and DQ/DT is $O(\epsilon)$.

Plane-wave solution

In the limit $\epsilon \to 0$ (2.15)–(2.17) with their right-hand sides replaced by zeros have a plane-wave solution. In this limit, the equations are ordinary differential equations in the phase θ, and may be integrated keeping $\mathbf{X}$ and T (i.e. N^2, R, $2\mathbf{\Omega}$, ω^* and $\boldsymbol{\varkappa}$) constant. The solution is

$$r = aN^2 \sin(\theta + \psi), \quad w = -a\omega^* \cos(\theta + \psi), \qquad (2.23), (2.24)$$

$$\mathbf{v} = w \frac{\kappa^2}{\kappa_H^2}\left(\mathbf{k} - \frac{n\boldsymbol{\varkappa}}{\kappa^2}\right) + \phi \frac{\boldsymbol{\varkappa} \times \mathbf{k}}{\kappa_H}, \qquad (2.25)$$

where

$$\phi = -a \frac{2\mathbf{\Omega} . \boldsymbol{\varkappa}}{\kappa_H} \sin(\theta + \psi), \qquad (2.26)$$

provided that

$$\omega^{*2} = (2\mathbf{\Omega} . \boldsymbol{\varkappa})^2/\kappa^2 + N^2\kappa_H^2/\kappa^2. \qquad (2.27)$$

Here $n = \boldsymbol{\varkappa} . \mathbf{k}$ is the vertical component of $\boldsymbol{\varkappa}$, κ is the magnitude of $\boldsymbol{\varkappa}$, and κ_H is the magnitude of $\boldsymbol{\varkappa}_H$, the horizontal component of $\boldsymbol{\varkappa}$. The solution contains an arbitrary amplitude a and phase ψ, both of which are functions of $\mathbf{X}$ and T. There is no restriction on the magnitude of a other than the requirement that the zero-order solution be $O(1)$ with respect to ϵ; this is a consequence of (2.15), which shows that the waves are transverse, and hence are sinusoidal even though nonlinear. The total phase $\theta + \psi$ may be regarded as equivalent to the expansion of Θ in powers of ϵ as $\theta + \psi = \epsilon^{-1}(\Theta + \epsilon\psi)$. Thus it would be legitimate to replace $\theta + \psi$ in (2.23) etc. by θ, and simultaneously expand Θ in powers of ϵ; however this would lead to extra terms in I, $\mathbf{F}$ and H. Here we shall retain $\theta + \psi$ in (2.23), etc., and assume that Θ has no explicit ϵ dependence.

In (2.23), etc., the amplitude a is that of the vertical displacement of a fluid particle. Equation (2.27) is the familiar dispersion relation for internal gravity waves in a rotating fluid (Phillips 1966, p. 193), and the corresponding group velocity is

$$\mathbf{c} = \nabla_\kappa \omega^* = -\frac{N^2 n}{\omega^* \kappa^2}\left(\mathbf{k} - \frac{n\boldsymbol{\varkappa}}{\kappa^2}\right) + \frac{2\mathbf{\Omega} . \boldsymbol{\varkappa}}{\omega^* \kappa^2}\left(2\mathbf{\Omega} - \frac{(2\mathbf{\Omega} . \boldsymbol{\varkappa})\boldsymbol{\varkappa}}{\kappa^2}\right). \qquad (2.28)$$

The group velocity is perpendicular to the phase velocity (parallel to $\boldsymbol{\varkappa}$), which is a consequence of the transverse nature of the waves. The wave energy density is defined to be

$$\mathscr{E} = \tfrac{1}{2}R(\langle |\mathbf{v}|^2 \rangle + (N^2)^{-1}\langle r^2 \rangle) = \tfrac{1}{2}Ra^2(\omega^*\kappa/\kappa_H)^2 \qquad (2.29)$$

and the wave action density is defined to be

$$\mathscr{F} = \mathscr{E}/\omega^*. \qquad (2.30)$$

$O(\epsilon)$ solution

At this stage, the amplitude a and phase ψ are undetermined. Equations governing a and ψ will now be derived through the requirement that the first-order solution for $\mathbf{v}$, r and q be periodic with period 2π and zero mean. It is to be understood that the variables r, etc., are expanded in powers of ϵ, e.g. $r = r_0 + \epsilon r_1 + \dots$, etc.; the plane-wave solution (2.23)–(2.25) corresponds to the subscript zero, while the equations for the variables r_1, etc., are (2.15)–(2.17), the variables on the left-hand sides of the equations being r_1, etc., while on the right-hand sides of the equations, I, $\mathbf{F}$ and H are evaluated using the plane-wave variables. However, we shall not display these subscripts in order to avoid excessive notation. Eliminating r, q and $\mathbf{v}_H$ from (2.15)–(2.17) yields the equation

$$w_{\theta\theta}+w = \epsilon J, \tag{2.31}$$

where
$$J = \frac{\mathbf{F}_\theta}{\omega^*}.\left(\mathbf{k}-\frac{n\boldsymbol{\kappa}}{\kappa^2}\right)-\frac{\mathbf{F}.(\boldsymbol{\kappa}\times\mathbf{k})}{\omega^{*2}\kappa^2}\,2\boldsymbol{\Omega}.\boldsymbol{\kappa} + \frac{\kappa_H^2 H}{\omega^{*2}\kappa^2}-\frac{nI_{\theta\theta}}{\kappa^2}-\frac{I_\theta}{\omega^*}\frac{2\boldsymbol{\Omega}.\boldsymbol{\kappa}\times\mathbf{k}}{\kappa^2}-\frac{I(2\Omega_V)(2\boldsymbol{\Omega}.\boldsymbol{\kappa})}{\omega^*\kappa^2}. \tag{2.32}$$

The general solution of (2.31) consists of the complementary function, which will be periodic and of zero mean, plus a particular integral given by

$$w = \epsilon\sin(\theta+\psi)\int_0^\theta J(\theta')\cos(\theta'+\psi)\,d\theta'-\epsilon\cos(\theta+\psi)\int_0^\theta J(\theta')\sin(\theta'+\psi)\,d\theta'. \tag{2.33}$$

This is periodic when
$$\langle J\sin(\theta+\psi)\rangle = \langle J\cos(\theta+\psi)\rangle = 0. \tag{2.34}$$

Also, when (2.34) is satisfied, it may be verified that (2.33) has zero mean (note that J itself is periodic with zero mean). Finally, it follows easily from (2.15)–(2.17) that, once w has been constructed to be periodic with zero mean, then $\mathbf{v}_H$, r and q are also periodic with zero mean. Thus conditions (2.33) are necessary and sufficient for the first-order solution to be periodic with zero mean (i.e. the asymptotic solution is well ordered).

Recalling the zero-order solutions for $\mathbf{v}$, r and q [i.e. (2.23)–(2.26)], the conditions (2.34) may be rewritten, after some manipulation, as

$$\langle \mathbf{v}.\mathbf{F}+N^{-2}rH+R^{-1}qI_\theta\rangle = 0, \tag{2.35a}$$

$$\langle \mathbf{v}_\theta.\mathbf{F}+N^{-2}r_\theta H+R^{-1}q_\theta I_\theta\rangle = 0. \tag{2.35b}$$

It is shown in appendix A that (2.35a) is just the equation for conservation of wave action:
$$\mathscr{F}_T+\nabla.[\mathscr{F}(\mathbf{c}+\mathbf{V})]+\lambda\kappa^2\mathscr{F} = 0, \tag{2.36}$$

where the dissipation coefficient λ is defined by (A 9). The conservation of wave action was established in the absence of rotation by Grimshaw (1974), and for the linearized case including rotation by Grimshaw (1975b); it has been established for a variety of other (linearized) physical systems by Bretherton & Garrett (1969). Equation (2.36) may be regarded as the required equation for the amplitude. Equation (2.35b) yields an equation for the phase ψ, which will not be displayed here as ψ does not appear explicitly in any other averaged equation.

3. Transport equations: reformulation using the Lagrangian mean velocity

The equations governing the mean flow variables $\mathbf{V}$, R and Q and their interaction with the wave amplitude are (2.12)–(2.14) and (2.36). They are displayed again here for convenience:

$$DR/DT + R\nabla.\mathbf{V} + \epsilon\nabla.\langle Rr\mathbf{v}\rangle = 0, \tag{3.1}$$

$$\epsilon R\, D\mathbf{V}/DT + R2\mathbf{\Omega}\times\mathbf{V} + \epsilon\nabla.\langle R\mathbf{v}\mathbf{v}\rangle + \epsilon 2\mathbf{\Omega}\times\langle Rr\mathbf{v}\rangle + \epsilon^{-1}\nabla Q + \epsilon^{-1}R\mathbf{k} = O(\epsilon^2), \tag{3.2}$$

$$\frac{DR}{DT} + \epsilon\nabla.\langle Rr\mathbf{v}\rangle - \frac{F}{\epsilon C^2}\frac{DQ}{DT} = O(\epsilon^2), \tag{3.3}$$

$$D\mathscr{F}/DT + \nabla.(\mathbf{c}\mathscr{F}) + \lambda\kappa^2\mathscr{F} = O(\epsilon). \tag{3.4}$$

These are supplemented by the dispersion relation (2.27), which, by virtue of (2.4), is a partial differential equation for the phase Θ. Here

$$D/DT \equiv \partial/\partial T + \mathbf{V}.\nabla \tag{3.5}$$

and is the time derivative in a frame moving at the Eulerian mean velocity $\mathbf{V}$. These equations show that the interaction of the waves with the mean flow cannot be entirely attributed to the Reynolds stress tensor $\langle R\mathbf{v}\mathbf{v}\rangle$, as there is also a contribution from the buoyancy flux $\langle Rr\mathbf{v}\rangle$; note that this buoyancy flux appears both as a mass source in (3.1) and also as an excess vortex force in (3.2). This has the consequence that the effect of the waves on the Eulerian mean flow cannot, in general, be expressed as the divergence of an appropriate 'radiation stress' tensor. However, as shown by Bretherton (1971) for certain linearized systems,† the introduction of a Lagrangian mean velocity will remove this conceptual difficulty and simplify (3.1)–(3.3).

The Lagrangian mean velocity $\mathbf{V}_L$ is defined such that an observer moving with velocity $\mathbf{V}_L$ records a zero mean for the particle displacements (relative to himself). If $\boldsymbol{\xi}$ denotes this particle displacement, it is shown in appendix B [(B 21) and (B 19)] that

$$\mathbf{V}_L = \mathbf{v} + \epsilon\{\langle r\mathbf{v}\rangle + R^{-1}\nabla.\langle R\boldsymbol{\xi}\mathbf{v}\rangle\} + O(\epsilon^2), \tag{3.6}$$

where
$$-\omega^*\boldsymbol{\xi}_\theta = \mathbf{v} + O(\epsilon). \tag{3.7}$$

Hence
$$\nabla.(R\mathbf{V}_L) = \nabla.(R\mathbf{V} + \epsilon\langle Rr\mathbf{v}\rangle) + O(\epsilon^2), \tag{3.8}$$

there being no contribution from the term $\langle R\boldsymbol{\xi}\mathbf{v}\rangle$, as it is an antisymmetric tensor. Thus (3.1) becomes

$$D_L R/DT + R\nabla.\mathbf{V}_L = O(\epsilon^2), \tag{3.9}$$

where
$$D_L/DT \equiv \partial/\partial T + \mathbf{V}_L.\nabla. \tag{3.10}$$

† Dr M. E. McIntyre (private communication) has also used Lagrangian mean velocities to discuss 'radiation stress' concepts for internal gravity waves in a rotating fluid, although he used the linearized equations of a Boussinesq fluid. See also McIntyre (1973) for another situation involving internal gravity waves in which a buoyancy flux term contributes to the mean flow, although on that occasion, the use of a Lagrangian mean velocity did not lead to a simple 'radiation stress' concept.

Similarly, substitution of (3.6) into (3.3) gives

$$\frac{D_L R}{DT} - \frac{F}{\epsilon C^2}\frac{D_L Q}{DT} = -\epsilon\nabla . \langle R r \mathbf{v}\rangle - N^2 \nabla . \langle R \boldsymbol{\xi} w\rangle + O(\epsilon^2)$$
$$= O(\epsilon^2), \tag{3.11}$$

on using (2.17) and (3.7). Thus the use of $\mathbf{V}_L$ in place of $\mathbf{V}$ has removed the buoyancy flux term from both (3.1) and (3.3). Also, in (3.4), D/DT may be replaced by D_L/DT, as the error in so doing is $O(\epsilon)$, and can be absorbed into the right-hand side. Similarly, in (2.10) [and hence in (2.4) and (2.27)], $\mathbf{V}$ may be replaced by $\mathbf{V}_L$ with an error $O(\epsilon)$. It remains to examine the effect of (3.6) on the mean momentum equation (3.2).

From (2.16) it follows that

$$\langle R\mathbf{v}\mathbf{v}\rangle = \langle q\mathbf{v}\boldsymbol{\kappa}/\omega^*\rangle - R\langle r_\theta \mathbf{v}/\omega^*\rangle \mathbf{k} - R\langle \mathbf{v}(2\boldsymbol{\Omega}\times\mathbf{v}_\theta)/\omega^*\rangle, \tag{3.12}$$

and using (3.7) and (A 6),

$$\langle R\mathbf{v}\mathbf{v}\rangle = \mathscr{F}\mathbf{c}\boldsymbol{\kappa} + R\langle r\boldsymbol{\xi}\rangle\mathbf{k} - R\langle \mathbf{v}(2\boldsymbol{\Omega}\times\boldsymbol{\xi})\rangle. \tag{3.13}$$

Substituting (3.6) and (3.13) into (3.2) gives, after some manipulation,

$$\epsilon R\, D_L \mathbf{V}_L/DT + R2\boldsymbol{\Omega}\times\mathbf{V}_L + \epsilon^{-1}\nabla Q + \epsilon^{-1}R\mathbf{k} + \epsilon\nabla . (\mathscr{F}\mathbf{c}\boldsymbol{\kappa}) + \epsilon\nabla . \langle R r\boldsymbol{\xi}\rangle\,\mathbf{k} = O(\epsilon^2). \tag{3.14}$$

Also, $$\langle R r\boldsymbol{\xi}\rangle = (\mathscr{F}N^2/\omega^*)(\mathbf{k} - n\boldsymbol{\kappa}/\kappa^2). \tag{3.15}$$

Substituting (3.16)–(3.18) into (3.14) gives

$$\epsilon R\, D_L \mathbf{V}_L/DT + R2\boldsymbol{\Omega}\times\mathbf{V}_L + \epsilon^{-1}\nabla Q + \epsilon^{-1}\mathbf{k}\{R + \epsilon^2\nabla . [\mathscr{F}N^2\omega^{*-1}(\mathbf{k} - n\boldsymbol{\kappa}/\kappa^2)]\}$$
$$+ \epsilon\nabla . (\mathscr{F}\mathbf{c}\boldsymbol{\kappa}) = O(\epsilon^2). \tag{3.16}$$

Thus the Reynolds stress term $\langle R\mathbf{v}\mathbf{v}\rangle$ and the buoyancy flux $\langle Rr\mathbf{v}\rangle$ have effectively combined to produce a 'radiation stress' tensor $\mathscr{F}\mathbf{c}\boldsymbol{\kappa}$, acting on the Lagrangian mean momentum equation; (3.16) has the same form as the result obtained by Grimshaw (1974) in the absence of rotation. Note that the tensor $\mathscr{F}\mathbf{c}\boldsymbol{\kappa}$ arises from the first term on the right-hand side of (3.12), which may be rewritten as $\langle \boldsymbol{\xi}\boldsymbol{\kappa} q_\theta\rangle$; thus the 'radiation stress' tensor arises from the average of the particle displacement with the pressure gradient. The terms in (3.16) of $O(\epsilon)$ and parallel to $\mathbf{k}$ determine an $O(\epsilon^2)$ change in the mean density R.

We shall complete this section with an analysis of the 'Stokes drift' velocity $\mathbf{V}_S$, which is just the difference between $\mathbf{V}_L$ and $\mathbf{V}$. Thus, from (3.6), it follows that

$$\mathbf{V}_S = \mathbf{V}_L - \mathbf{V} = \epsilon\langle r\mathbf{v}\rangle + \epsilon R^{-1}\nabla . \langle R\boldsymbol{\xi}\mathbf{v}\rangle + O(\epsilon^2). \tag{3.17}$$

But it may be shown that

$$\langle r\mathbf{v}\rangle = -N^2\langle w\boldsymbol{\xi}\rangle, \tag{3.18}$$

and hence $$\mathbf{V}_S = (F/C^2)\langle w\boldsymbol{\xi}\rangle + \epsilon\nabla . \langle \boldsymbol{\xi}\mathbf{v}\rangle + O(\epsilon^2). \tag{3.19}$$

Explicit calculation now shows that

$$\mathbf{V}_S = \frac{F}{C^2}\frac{\mathscr{F}(2\boldsymbol{\Omega} . \boldsymbol{\kappa})\,\boldsymbol{\kappa}\times\mathbf{k}}{R\omega^*\kappa^2} - \epsilon\boldsymbol{\kappa}\times\nabla\left(\frac{\mathscr{F}(2\boldsymbol{\Omega} . \boldsymbol{\kappa})}{R\omega^*\kappa^2}\right) + O(\epsilon^2). \tag{3.20}$$

The first term is due to the first-order effects of compressibility, and is non-zero even for plane waves (i.e. unmodulated waves). The second term is due primarily to the modulation of the wave amplitude, and will normally be most significant at the extremities of a wave packet (where the gradients of $\mathscr{F}$ are strongest). In the absence of rotation, $\mathbf{V}_S$ is $O(\epsilon^2)$.

4. Transport equations: vorticity considerations

The transport equations are now (3.9), (3.16), (3.11) and (3.4), together with the dispersion relation (2.27). For convenience, these are displayed here:

$$D_L R/DT + R\nabla . \mathbf{V}_L = O(\epsilon^2), \tag{4.1}$$

$$\epsilon R\frac{D_L\mathbf{V}_L}{DT} + R2\mathbf{\Omega}\times\mathbf{V}_L + \frac{\nabla Q}{\epsilon} + \frac{\mathbf{k}}{\epsilon}\left\{R + \epsilon^2\nabla . \left[\frac{\mathscr{F}N^2}{\omega^*}\left(\mathbf{k} - \frac{n\boldsymbol{\varkappa}}{\kappa^2}\right)\right]\right\} + \epsilon\nabla . (\mathscr{F}\mathbf{c}\boldsymbol{\varkappa}) = O(\epsilon^2). \tag{4.2}$$

$$\frac{D_L R}{DT} - \frac{F}{\epsilon C^2}\frac{D_L Q}{DT} = O(\epsilon^2) \tag{4.3}$$

$$D_L\mathscr{F}/DT + \nabla . (\mathscr{F}\mathbf{c}) + \lambda\kappa^2\mathscr{F} = O(\epsilon), \tag{4.4}$$

$$\kappa^2\omega^{*2} = (2\mathbf{\Omega} . \boldsymbol{\varkappa})^2 + N^2\kappa_H^2, \tag{4.5}$$

$$\omega^* = \omega - \boldsymbol{\varkappa} . \mathbf{V}_L + O(\epsilon). \tag{4.6}$$

Equation (4.2) shows that, to $O(\epsilon)$, the mean flow is in hydrostatic and geostrophic balance; the parameter ϵ plays the role of a Rossby number for the mean flow. (Note that, since $2\mathbf{\Omega}$ and N have been scaled by the same factor, N_1 may be also regarded as a typical value of the angular velocity; also it has been assumed that g/N_1^2 is a length scale for the mean flow, and the velocity scale is $N_1 L$, thus $\epsilon = N_1^2 L/g$ may be regarded as the ratio of the velocity scale to the product of the large length scale and the angular velocity.) The familiar geostrophic relations now hold. Thus the vertical component of (4.2) determines R and the horizontal component of (4.2) shows that, since $\mathbf{V}_L$ is horizontal to $O(\epsilon)$,

$$2\Omega_V R\mathbf{V}_L = \mathbf{k}\times\nabla_H Q/\epsilon + O(\epsilon). \tag{4.7}$$

Equation (4.7) shows that $\nabla_H Q$ is $O(\epsilon)$. Eliminating the pressure from (4.2) shows that

$$\nabla R\times\mathbf{k} = -\epsilon\,\mathrm{curl}\,(2\mathbf{\Omega}\times\mathbf{V}_L) + O(\epsilon^2). \tag{4.8}$$

The horizontal component of (4.8) implies that

$$\nabla_H R = \epsilon 2\mathbf{\Omega} . \nabla(\mathbf{k}\times R\mathbf{V}) + O(\epsilon^2), \tag{4.9}$$

and so $\nabla_H R$ is $O(\epsilon)$. This is just the thermal-wind equation of meteorology. Equation (4.1) shows that $\mathbf{V}_L$ is horizontally non-divergent to $O(\epsilon)$, and from (4.7), $-Q/\epsilon 2\Omega_V R$ acts as a stream function for $\mathbf{V}_L$. Finally the left-hand side of (4.3) is $O(\epsilon)$ and determines the vertical velocity W_L:

$$RN^2 W_L = R_T - (F/\epsilon C^2)\,Q_T + \mathbf{V}_L . \nabla_H R + O(\epsilon^2). \tag{4.10}$$

To determine the time variation of $\mathbf{V}_L$ we must now examine the $O(\epsilon)$ terms in the mean momentum equation (4.2). This is a familiar procedure for quasi-geostrophic systems, and the present situation may be regarded as a quasi-geostrophic system with a forcing term in the momentum equation due to the waves. Indeed the effect of the waves on the mean flow is due entirely to the divergence of the 'radiation stress' tensor $\mathscr{F}\mathbf{c}\boldsymbol{\kappa}$ in the mean momentum equation (4.2), together with a feedback mechanism by virtue of (4.6) and (4.4). First, using (4.4) and the compatibility relation (A 11) between $\boldsymbol{\kappa}$ and ω, it may be shown that

$$\nabla.(\mathscr{F}\mathbf{c}\boldsymbol{\kappa})+\mathscr{F}\nabla_e\omega^* = -D_L(\boldsymbol{\kappa}\mathscr{F})/DT-\mathscr{F}\nabla\mathbf{V}_L.\boldsymbol{\kappa}-\lambda\kappa^2\boldsymbol{\kappa}\mathscr{F}, \tag{4.11}$$

where $\nabla_e\omega^*$ is the explicit derivative of ω^* with respect to $\mathbf{X}$, keeping $\boldsymbol{\kappa}$ fixed (parallel to $\mathbf{k}$ here). Let

$$\mathbf{U} = R^{-1}\mathscr{F}\boldsymbol{\kappa}, \tag{4.12}$$

where $R\mathbf{U}$ has the dimensions of momentum and in linear theories is sometimes referred to as 'wave momentum'; (4.2) becomes

$$\epsilon\frac{D_L\mathbf{V}_L}{DT}+2\boldsymbol{\Omega}\times\mathbf{V}_L+\frac{\nabla Q}{\epsilon R}+\frac{\mathbf{k}}{\epsilon}\left\{1+\epsilon^2\frac{1}{R}\nabla.\left[\frac{\mathscr{F}N^2}{\omega^*}\left(\mathbf{k}-\frac{n\boldsymbol{\kappa}}{\kappa^2}\right)\right]+\frac{(N^2)_z\kappa_H^2}{2\omega^*\kappa^2}\mathbf{k}\right\}$$
$$= \epsilon\left(\frac{D_L\mathbf{U}}{DT}+\nabla\mathbf{V}_L.\mathbf{U}+\lambda\kappa^2\mathbf{U}\right)+O(\epsilon^2). \tag{4.13}$$

We shall now examine the nature of the forcing term on the right-hand side of (4.13) by, first, establishing a circulation theorem and, second, deriving a potential-vorticity equation.

Let $\mathscr{C}$ be a circuit which moves with the mean velocity $\mathbf{V}_L$ ($\mathscr{C}$ is horizontal to $O(\epsilon)$]. Then

$$\frac{D}{DT}\oint_{\mathscr{C}}\mathbf{V}_L.d\mathbf{X} = \oint_{\mathscr{C}}\frac{D\mathbf{V}_L}{DT}.d\mathbf{X}$$
$$= \oint_{\mathscr{C}}\left\{\frac{D_L\mathbf{U}}{DT}+\nabla\mathbf{V}_L.\mathbf{U}+\lambda\kappa^2\mathbf{U}-\frac{\nabla Q}{\epsilon^2R}-\frac{2\boldsymbol{\Omega}\times\mathbf{V}_L}{\epsilon}\right\}.d\mathbf{X}+O(\epsilon). \tag{4.14}$$

Now it may be shown that

$$\oint_{\mathscr{C}}2\boldsymbol{\Omega}\times\mathbf{V}_L.d\mathbf{X} = \frac{D}{DT}\oint_{\mathscr{C}}\boldsymbol{\Omega}\times\mathbf{X}.d\mathbf{X} = \frac{D}{DT}\iint_{\mathscr{S}}2\boldsymbol{\Omega}.\mathbf{n}\,dS, \tag{4.15}$$

where $\mathscr{S}$ is any material surface whose boundary is $\mathscr{C}$ and $\mathbf{n}$ is the normal to $\mathscr{S}$. Also, it may be shown that

$$\frac{D}{DT}\oint_{\mathscr{C}}\mathbf{U}.d\mathbf{X} = \oint_{\mathscr{C}}\left\{\frac{D_L\mathbf{U}}{DT}+\nabla\mathbf{V}_L.\mathbf{U}\right\}.d\mathbf{X}. \tag{4.16}$$

Thus the circulation theorem, in the present context, is

$$\frac{D}{DT}\left\{\oint_{\mathscr{C}}\mathbf{V}.d\mathbf{X}+\iint_{\mathscr{S}}\frac{2\boldsymbol{\Omega}.\mathbf{n}\,dS}{\epsilon}\right\} = \frac{D}{DT}\oint_{\mathscr{C}}\mathbf{U}.d\mathbf{X}+\oint_{\mathscr{C}}\lambda\kappa^2\mathbf{U}.d\mathbf{X}-\oint_{\mathscr{C}}\frac{\nabla Q}{\epsilon^2R}.d\mathbf{X}. \tag{4.17}$$

The last term in (4.17) is $O(1)$. Indeed, it may be written in the form

$$-\oint_{\mathscr{C}} \frac{\nabla Q . d\mathbf{X}}{\epsilon^2 R} = \iint_{\mathscr{S}} \frac{\nabla Q \times \nabla R}{\epsilon^2 R^2} . \mathbf{n}\, dS. \tag{4.18}$$

A lengthy calculation then shows that

$$\frac{\nabla Q \times \nabla R}{\epsilon^2 R^2} = \frac{(2\mathbf{\Omega}.\nabla)\,\mathbf{V}_{LH}}{\epsilon} + \{2\mathbf{\Omega} \times \mathbf{V}_L . \operatorname{curl}(2\mathbf{\Omega} \times \mathbf{V}_L)\}\,\mathbf{k} + O(\epsilon). \tag{4.19}$$

Here $\mathbf{V}_{LH}$ is the horizontal component of $\mathbf{V}_L$. Since $\mathbf{n} = \mathbf{k} + O(\epsilon)$, substitution of (4.19) into (4.18) shows that this term is $O(1)$. Of the remaining two terms on the right-hand side of (4.17), the first describes the instantaneous production of mean vorticity in the vicinity of a wave packet due to $\mathbf{U}$, and the second describes the permanent production of mean vorticity by frictional effects associated with the passage of a wave packet.

In quasi-geostrophic systems, the customary procedure is to expand all variables in powers of ϵ:

$$Q = Q_0 + \epsilon Q_1 + \dots, \quad R = R_0 + \epsilon R_1 + \dots, \quad \mathbf{V}_L = \mathbf{V}_0 + \epsilon \mathbf{V}_1 + \dots. \tag{4.20}$$

Then Q_0 and R_0 are functions of z alone, and $Q_{0z} = -R_0$. To the next order in ϵ, the velocity $\mathbf{V}_0$ is determined by the geostrophic equation (4.7) (with a zero subscript on the left-hand side and Q_1 appearing on the right-hand side). The time variation of $\mathbf{V}_0$ is then determined by eliminating $\mathbf{V}_1$ from (4.1) and (4.2). However, it is well known that this procedure generates the potential-vorticity equation, and we shall proceed instead by deriving the potential-vorticity equation directly. Thus, taking the curl of (4.41), we obtain the vorticity equation

$$\frac{D_L}{DT}\left\{\frac{\epsilon \operatorname{curl} \mathbf{V}_L + 2\mathbf{\Omega}}{R}\right\} = \frac{(\epsilon \operatorname{curl} \mathbf{V}_L + 2\mathbf{\Omega})}{R} . \nabla \mathbf{V}_L + \frac{\nabla R \times \nabla Q}{\epsilon R^3} + \frac{\epsilon}{R} \operatorname{curl} \mathbf{F} + O(\epsilon^2),$$

where $$\mathbf{F} = \frac{D_L \mathbf{U}}{DT} + \nabla \mathbf{V}_L . \mathbf{U} + \lambda \kappa^2 \mathbf{U} - \mathbf{k} \frac{1}{R} \nabla . \left(\frac{\mathscr{F} N^2}{\omega^*} (\mathbf{k} - n\boldsymbol{\kappa}/\kappa^2)\right). \tag{4.21}$$

Ertel's theorem states that, if

$$D_L \chi / DT = \gamma, \tag{4.22}$$

then (cf. Pedlosky 1971)

$$\frac{D_L}{DT}\left\{\frac{(\epsilon \operatorname{curl} \mathbf{V}_L + 2\mathbf{\Omega}) . \nabla \chi}{R}\right\} = \nabla \chi \left\{\frac{\epsilon \operatorname{curl} \mathbf{F}}{R} + \frac{\nabla R \times \nabla Q}{\epsilon R^3}\right\} + \frac{\epsilon \operatorname{curl} \mathbf{V}_L + 2\mathbf{\Omega}}{R} . \nabla \gamma + O(\epsilon^2). \tag{4.23}$$

The potential-vorticity equation is now generated by choosing χ to be a function of R and Q (so that $\nabla \chi . \nabla R \times \nabla Q$ is zero) for which γ is $O(\epsilon^2)$. An appropriate choice is $\chi = S$, where $S = S(R, Q)$ is the entropy associated with the mean flow. Then

$$\frac{\partial S}{\partial R} = -\frac{A}{R}, \quad \frac{\partial S}{\partial Q} = A \frac{F}{\epsilon R C^2}, \tag{4.24}$$

where A is a thermodynamic coefficient, and is the ratio of the specific heat at constant pressure to the product of the temperature and the coefficient of thermal expansion (A is constant for an ideal gas). Hence, from (4.3)

$$D_L S/DT = O(\epsilon^2). \tag{4.25}$$

Also
$$\frac{\nabla S}{A} = N^2\mathbf{k} - \frac{F}{C^2}2\mathbf{\Omega}\times\mathbf{V}_L - \frac{\nabla_H R}{R} + O(\epsilon^2). \tag{4.26}$$

Substitution of $\chi = S$ into (4.23) then gives

$$\frac{D_L}{DT}\left\{\frac{AN^2}{R}(\mathbf{k}\,.\operatorname{curl}\mathbf{V}_L) - \frac{2\mathbf{\Omega}\,.\nabla_H R}{\epsilon R^2}\right\} = \frac{AN^2}{R}\mathbf{k}\,.\operatorname{curl}\mathbf{F} + O(\epsilon). \tag{4.27}$$

It may be shown that

$$\mathbf{k}\,.\operatorname{curl}\mathbf{F} = D_L\{\mathbf{k}\,.\operatorname{curl}\mathbf{U}\}/DT + \mathbf{k}\,.\operatorname{curl}(\lambda\kappa^2\mathbf{U}) + O(\epsilon). \tag{4.28}$$

A lengthy calculation of the left-hand side of (4.27) then gives the result

$$\frac{D_{LH}}{DT}\left\{\nabla_H\cdot\left(\frac{\nabla_H Q_1}{2\Omega_V R_0}\right) + \frac{1}{R_0}2\mathbf{\Omega}\,.\nabla\left(\frac{R_0}{N_0^2 2\Omega_V}2\mathbf{\Omega}\,.\nabla\left(\frac{Q_1}{R_0}\right)\right)\right\}$$
$$= \frac{D_{LH}}{DT}\{\mathbf{k}\,.\operatorname{curl}\mathbf{U}\} + \mathbf{k}\,.\operatorname{curl}(\lambda\kappa^2\mathbf{U}) + O(\epsilon), \tag{4.29}$$

where
$$D_{LH}/DT \equiv \partial/\partial T + \mathbf{V}_L\,.\nabla_H \tag{4.30}$$

and
$$2\Omega_V R_0\mathbf{V}_L = \mathbf{k}\times\nabla_H Q_1 + O(\epsilon). \tag{4.31}$$

Here the subscripts zero denote quantities evaluated at (R_0, Q_0), which are functions of Z alone. The quantity in the brackets on the left-hand side of (4.31) is the potential vorticity, and the equation thus describes the generation of potential vorticity by the waves.

Appendix A. Derivation of the wave action equation

In § 2 it was shown that the equations governing the wave amplitude and phase are (2.35), i.e.

$$\langle\mathbf{v}\,.\mathbf{F} + N^{-2}rH + R^{-1}qI_\theta\rangle = 0, \tag{A 1}$$

$$\langle\mathbf{v}_\theta\,.\mathbf{F} + N^{-2}r_\theta H + R^{-1}q_\theta I_\theta\rangle = 0. \tag{A 2}$$

Here I_θ, $\mathbf{F}$ and H are given by (2.20), (2.21) and (2.22) respectively, and $\mathbf{v}$ and r are given by (2.23)–(2.26) [q is then given by (2.16)]. Thus substituting the expressions for I_θ, $\mathbf{F}$ and H into (A 1) gives

$$\Big\langle\mathbf{v}\,.D\mathbf{v}/DT + \mathbf{v}\,.\{\mathbf{v}\,.\nabla\mathbf{V}\} + r\mathbf{v}\,.2\mathbf{\Omega}\times\mathbf{V} + R^{-1}\mathbf{v}\,.\nabla q$$
$$+ N^{-2}\left\{r\frac{Dr}{DT} + \frac{r\mathbf{v}_H\,.\nabla_H R}{\epsilon R} + \frac{F}{\epsilon}\frac{1}{RC^2}r\mathbf{v}\,.2\mathbf{\Omega}\times\mathbf{V} + \frac{F}{\epsilon}\frac{\omega^* r q_\theta}{RC^2}\right\}$$
$$+ R^{-1}q\nabla\,.\mathbf{v} - R^{-1}\frac{F}{\epsilon}\frac{wq_\theta}{C^2}\Big\rangle + \epsilon^{-1}E\kappa^2\langle\bar{\nu}\,|\mathbf{v}|^2 + \sigma\bar{k}r^2\rangle = O(\epsilon). \tag{A 3}$$

Here $\bar{\nu} = \bar{\mu}/R$ is the kinematic viscosity evaluated at (Q, R). Then, using (2.15)–(2.17) (with the right-hand sides $O(\epsilon)$), which relate the variables $\mathbf{v}$, r and q, it follows that

$$\frac{D}{DT}\langle \tfrac{1}{2}|\mathbf{v}|^2 + \tfrac{1}{2}r^2/N^2\rangle + R^{-1}\nabla.\langle q\mathbf{v}\rangle + \frac{\boldsymbol{\varkappa}.\{\langle q\mathbf{v}\rangle.\nabla\mathbf{V}\}}{\omega^* R} - \frac{\langle 2\boldsymbol{\Omega}\times\mathbf{v}_\theta.\{\mathbf{v}.\nabla\mathbf{V}\}\rangle}{\omega^*}$$
$$+ \frac{\langle r\mathbf{v}_H\rangle.\nabla_H R}{\epsilon R N^2} - \frac{R_z}{N^2 R}\langle r\mathbf{v}\rangle.2\boldsymbol{\Omega}\times\mathbf{V} + \epsilon^{-1}E\kappa^2\langle\bar{\nu}|\mathbf{v}|^2 + \sigma\bar{k}r^2\rangle = O(\epsilon). \quad \text{(A 4)}$$

Now, from (2.29),

$$R\langle\tfrac{1}{2}|\mathbf{v}|^2 + \tfrac{1}{2}r^2/N^2\rangle = \mathscr{E}, \quad \text{(A 5)}$$

and explicit calculation shows that

$$\langle q\mathbf{v}\rangle = \mathscr{E}\mathbf{c}, \quad \text{(A 6)}$$

where $\mathbf{c}$ is the group velocity (2.28). Next, it may be shown that

$$\frac{\langle r\mathbf{v}_H\rangle.\nabla_H R}{\epsilon R N^2} - \frac{R_z}{N^2 R}\langle r\mathbf{v}\rangle.2\boldsymbol{\Omega}\times\mathbf{V} - \frac{\langle 2\boldsymbol{\Omega}\times\mathbf{v}_\theta.\{\mathbf{v}.\nabla\mathbf{V}\}\rangle}{\omega^*} = 0, \quad \text{(A 7)}$$

where use has been made of (2.13) to relate $\nabla_H R$ to $\mathbf{V}$. Also, explicit calculation shows that

$$\epsilon^{-1}E\kappa^2\langle\bar{\nu}|\mathbf{v}|^2 + \sigma\bar{k}r^2\rangle = R^{-1}\lambda\kappa^2\mathscr{E}, \quad \text{(A 8)}$$

where

$$\lambda = \frac{E}{\epsilon}\left[\bar{\nu} + \sigma\bar{k} + \frac{(2\boldsymbol{\Omega}.\boldsymbol{\varkappa})^2(\bar{\nu} - \sigma\bar{k})}{\{(2\boldsymbol{\Omega}.\boldsymbol{\varkappa})^2 + N^2\kappa_H^2\}}\right]. \quad \text{(A 9)}$$

Substitution of these results into (A 4) gives

$$D\mathscr{E}/DT + \nabla.(\mathscr{E}\mathbf{c}) + \mathscr{E}\boldsymbol{\varkappa}.\{\mathbf{c}.\nabla\mathbf{V}\}/\omega^* + \lambda\kappa^2\mathscr{E} = O(\epsilon). \quad \text{(A 10)}$$

Now it follows from (2.4) that

$$\boldsymbol{\varkappa}_T + \nabla\omega = 0, \quad \text{(A 11)}$$

and substituting $\omega = \omega^* + \boldsymbol{\varkappa}.\mathbf{V}$ into this equation, where ω^* is given by the dispersion relation (2.27), we have

$$D\omega^*/DT + \mathbf{c}.\nabla\omega^* = -\boldsymbol{\varkappa}.\{\mathbf{c}.\nabla\mathbf{V}\}. \quad \text{(A 12)}$$

(Note that (A 12) should also contain, on the right-hand side, a term $(D\omega^*/DT)_e$, the explicit derivative of ω^* with respect to $(\mathbf{X}, T)$, keeping $\boldsymbol{\varkappa}$ fixed; in the present context this is proportional to DN^2/DT and is $O(\epsilon)$. However, if Q and R are allowed to depend on T, as well as Z, to leading order in ϵ, terms proportional to DN^2/DT appear in (A 10) and (A 13) and it may be shown that the equation for wave action derived below is still valid. See Grimshaw (1975), where this case was considered for a linearized system.)

Substituting (A 12) into (A 10) gives

$$\frac{D\mathscr{E}}{DT} + \nabla.(\mathscr{E}\mathbf{c}) - \frac{\mathscr{E}}{\omega^*}\left\{\frac{D\omega^*}{DT} + \mathbf{c}.\nabla\omega^*\right\} + \lambda\kappa^2\mathscr{E} = O(\epsilon) \quad \text{(A 13)}$$

or recalling that $\mathscr{E} = \omega^*\mathscr{F}$,

$$\mathscr{F}_T + \nabla.[\mathscr{F}(\mathbf{c} + \mathbf{V})] + \lambda\kappa^2\mathscr{F} = O(\epsilon). \quad \text{(A 14)}$$

This is the equation for the conservation of wave action.

Equation (A 14), derived from (A 1), is an equation for the amplitude a, and does not contain the phase ψ; (A 2) leads to an equation for ψ. Indeed it may easily be shown, on substituting (2.23)–(2.26) into (A 2), that

$$D\psi/DT + \mathbf{c}.\nabla\psi + \ldots = 0, \tag{A 15}$$

where the omitted terms do not involve ψ and have not been displayed as they are rather complicated (these terms all involve $2\mathbf{\Omega}$ and are identically zero in the absence of rotation).

Appendix B. Derivation of the Lagrangian mean velocity

In order to relate the Eulerian mean velocity $\mathbf{V}$ to the Lagrangian mean velocity $\mathbf{V}_L$ we shall use a modified Lagrangian description of the motion (cf. Bretherton 1971). Let $\mathbf{X}_L$ be the position, at time T, of an observer moving with velocity $\mathbf{V}_L$; then $\mathbf{X}_L$ is the solution of the equation

$$D_L\mathbf{X}_L/DT = \mathbf{V}_L(\mathbf{X}_L, T), \tag{B 1}$$

where
$$D_L/DT \equiv \partial/\partial T + \mathbf{V}_L.\nabla_L \tag{B 2}$$

is the time derivative following $\mathbf{V}_L$. The subscript L denotes differentiation with respect to $\mathbf{X}_L$. The particle displacement, relative to $\mathbf{V}_L$, is $\boldsymbol{\xi}$, so that

$$\mathbf{X} = \mathbf{X}_L + \epsilon\boldsymbol{\xi}(\theta_L; \mathbf{X}_L, T; \epsilon), \tag{B 3}$$

where
$$\theta_L = \epsilon^{-1}\Theta_L(\mathbf{X}_L, T). \tag{B 4}$$

Here $\boldsymbol{\xi}$ is periodic, of period 2π, and has *zero mean* in the phase θ_L. (It will be shown subsequently that $\Theta_L(\mathbf{X}_L, T) \equiv \Theta(\mathbf{X}_L, T)$.) Recalling that $\mathbf{X} = \epsilon\mathbf{x}$, and letting $\mathbf{X}_L = \epsilon\mathbf{x}_L$, (B 3) becomes

$$\mathbf{x} = \mathbf{x}_L + \boldsymbol{\xi}. \tag{B 5}$$

The mean Lagrangian density is ρ_L, defined to be that density which is conserved by $\mathbf{V}_L$, so that

$$D_L\rho_L/DT + \rho_L\nabla_L.\mathbf{V}_L = 0. \tag{B 6}$$

The equation for conservation of mass is

$$\rho J = \rho_L, \tag{B 7}$$

where J is the Jacobian of the transformation from $\mathbf{x}_L$ to $\mathbf{x}$:

$$J = \frac{\partial(\mathbf{x})}{\partial(\mathbf{x}_L)} = \tfrac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\frac{\partial x_i}{\partial x_{L\alpha}}\frac{\partial x_j}{\partial x_{L\beta}}\frac{\partial x_k}{\partial x_{L\gamma}}. \tag{B 8}$$

Here ϵ_{ijk} is the permutation symbol and the summation convention applies. Substitution of (B 5) into (B 8) gives

$$J = 1 + \frac{\partial \xi_\alpha}{\partial x_{L\alpha}} + \tfrac{1}{2}\epsilon_{ijk}\epsilon_{i\beta\gamma}\frac{\partial \xi_j}{\partial x_{L\beta}}\frac{\partial \xi_k}{\partial x_{L\gamma}} + \frac{\partial(\boldsymbol{\xi})}{\partial(\mathbf{x}_L)}. \tag{B 9}$$

We now let
$$\rho = R_L(\mathbf{X}_L, T; \epsilon)(1 + \epsilon r_L(\theta_L; \mathbf{X}_L, T; \epsilon)), \tag{B 10}$$

where r_L is periodic and has zero mean in the phase θ_L. Substitution of (B 9) and (B 10) into (B 7), and subsequent lengthy calculation, shows that

$$\varkappa_L . \xi_{\theta_L} = -\epsilon\{\nabla_L . \xi + r_L + \xi . (\xi_{\theta_L} . \nabla_L \varkappa_L)\} + O(\epsilon^2), \tag{B 11}$$

where
$$\varkappa_L = \nabla_L \Theta_L, \quad \omega_L = -\Theta_{LT}. \tag{B 12}$$

The fact that $\varkappa_L . \xi_{\theta_L}$ is $O(\epsilon)$ is the Lagrangian counterpart of (2.15). It may also be shown that R_L differs from ρ_L by terms $O(\epsilon^2)$.

We are now in a position to examine the relation between Eulerian and Lagrangian means. The Eulerian phase, using (B 3) and (B 11), is given by

$$\left.\begin{aligned} \Theta(\mathbf{X}, T) &= \Theta(\mathbf{X}_L, T) + \epsilon^2\alpha, \\ \alpha_\theta &= -\nabla_L . \xi - r_L + O(\epsilon). \end{aligned}\right\} \tag{B 13}$$

where (as above)

The fact that $\Theta(\mathbf{X}, T)$ and $\Theta(\mathbf{X}_L, T)$ differ by terms $O(\epsilon^2)$ means that we may identify the Lagrangian phase $\Theta_L(\mathbf{X}_L, T)$ with $\Theta(\mathbf{X}_L, T)$; then

$$\theta = \theta_L + \epsilon\alpha,$$

and any function periodic in θ may be expressed in terms of a series of functions periodic in θ_L. Hence if $f(\theta; \mathbf{X}, T)$ is some quantity expressed in terms of Eulerian variables, we find that

$$f(\theta; \mathbf{X}, T) = f(\theta_L + \epsilon\alpha; \mathbf{X}_L + \epsilon\xi, T) = f(\theta_L, \mathbf{X}_L . T) + \epsilon(\alpha f_\theta + \xi . \nabla f) + O(\epsilon^2). \tag{B 14}$$

To leading order in ϵ, Lagrangian and Eulerian means agree, but there is an $O(\epsilon)$ difference, given by

$$\langle f\rangle_L = \langle f\rangle + \epsilon\langle \alpha f_\theta + \xi . \nabla f\rangle + O(\epsilon^2). \tag{B 15}$$

Here a subscript L denotes a Lagrangian mean. Applying the relation (B 14) to the density ρ shows that

$$r_L = r + \xi . \nabla R/R + O(\epsilon), \tag{B 16}$$

and R_L (the Lagrangian mean density) differs from R (the Eulerian mean density) only by terms $O(\epsilon^2)$; both differ from ρ_L by terms $O(\epsilon^2)$.

The velocity $\mathbf{u}$ is obtained by differentiating (B 3) with respect to the time:

$$\mathbf{u} = \mathbf{V}_L - \omega_L^* \xi_{\theta_L} + \epsilon D_L \xi/DT, \tag{B 17}$$

where
$$\omega_L^* = \omega_L - \varkappa_L . \mathbf{V}_L. \tag{B 18}$$

Since this is just $\mathbf{V} + \mathbf{v}$ (in Eulerian terms), it follows from (B 14) and (B 15) that

$$\mathbf{v} = -\omega_L^* \xi_{\theta_L} + O(\epsilon) \tag{B 19}$$

and
$$\mathbf{V}_L = \mathbf{V} + \epsilon\langle \alpha \mathbf{v}_\theta + \xi . \nabla \mathbf{v}\rangle + O(\epsilon^2). \tag{B 20}$$

Using (B 13), (B 16) and (B 19), this last result takes the form

$$\mathbf{V}_L = \mathbf{V} + \epsilon\langle r\mathbf{v}\rangle + \epsilon R^{-1}\nabla . \langle R\xi\mathbf{v}\rangle + O(\epsilon^2). \tag{B 21}$$

We conclude this appendix with a demonstration of how the equations (4.1)–(4.3) involving the Lagrangian mean velocity may be derived using Lagrangian means (cf. Bretherton 1971). The Lagrangian equations of motion are (B 6) and

(B 7) (conservation of mass), (1.6) (conservation of entropy) and (1.5) (conservation of momentum); (1.5) may be rewritten as

$$\rho_L \frac{d\mathbf{u}}{dt} + \rho_L 2\mathbf{\Omega} \times \mathbf{u} + \frac{\rho_L \mathbf{k}}{\epsilon} + \frac{J}{\epsilon^2} \nabla p = e E\mu \nabla^2 \mathbf{u} + \dots, \tag{B 22}$$

where

$$\frac{d}{dt} = -\omega^*_{\theta L} \frac{\partial}{\partial \theta_L} + \epsilon \frac{D_L}{DT}. \tag{B 23}$$

Since R differs from ρ_L only by terms $O(\epsilon^2)$, (4.1) is just (B 6); averaging (1.6) with respect to θ_L gives (4.3), and similarly averaging (B 22) gives

$$\epsilon \rho_L \frac{D\mathbf{V}_L}{DT} + \rho_L 2\mathbf{\Omega} \times \mathbf{V}_L + \frac{\rho_L \mathbf{k}}{\epsilon} + \epsilon^{-2} \langle J \nabla p \rangle_L = O(\epsilon^2). \tag{B 24}$$

Now it may be shown that

$$J\, \partial p / \partial x_i = \partial (C_{\alpha i} p) / \partial x_{L\alpha}, \tag{B 25}$$

where

$$C_{\alpha i} = \tfrac{1}{2} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} \frac{\partial x_j}{\partial x_{L\beta}} \frac{\partial x_k}{\partial x_{L\gamma}}. \tag{B 26}$$

Hence

$$\epsilon^{-2} \langle J\, \partial p / \partial x_i \rangle_L = \epsilon^{-1}\, \partial \langle c_{\alpha i} p \rangle / \partial X_{L\alpha}. \tag{B 27}$$

A lengthy calculation, which will not be reproduced here, then shows that (B 24) agrees with (4.2).

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